UNIT-IV

*** Orthogonal Complement:-** let M be a subset of a Hilbert space X then the orthogonal complement of M is the set of all element of X which are orthogonal to every element of M.

we write $M^\perp$ to denote the orthogonal complement of M. characteristics are given below:

(i) $M^\perp$ is closed subspace of X for every $M \subset X$.

(ii) It may consist of the zero vector only.

(iii) If $M \neq \{0\}$ then $M^\perp \neq X$.

9.10.2023

*** Projection Theorem:-**

Statement:- If M is a closed subspace of a Hilbert space X and $x \in X$ then $\exists$ a unique element y in M and $z \in M^\perp$ such that

$$x = y + z$$

i.e. $X = M + M^\perp$

Proof:- let

$$K = \{x-k : k \in M\}$$

we claim that, K is a non.empty closed and convex subset of X.

We Prove that K is closed:-

let $\{x - k_n\}$ be any sequence in K s.t. $x-k_n \to x-k$ as $n \to \infty$ where, $k, k_n \in M$, $x \in X$.

Thus,

$$\|k_n - k\| = \|x - k_n - x + k\|$$
$$= \|(x-k_n) - (x-k)\|$$
$$\to 0 \text{ as } n \to \infty$$

$\|k_n - k\| \to 0$ as $n \to \infty$

$\Rightarrow k_n \to k$ as $n \to \infty$

$\Rightarrow \{k_n\}$ be a sequence in M and M is closed so $k \in M$

Hence, $x - k \in K$.

$\Rightarrow$ K is closed.

To prove K is convex:- let two points $z_1, z_2 \in K$ then

$$z_1 = x - k_1$$

and $z_2 = x - k_2$

when $k_1, k_2 \in M$

if

$$\lambda z_1 + (1-\lambda) z_2 = \lambda(x-k_1) + (1-\lambda)(x-k_2)$$
$$= (\lambda x - \lambda k_1) + (x - k_2 - \lambda x + \lambda k_2)$$
$$= \lambda x - \lambda k_1 + x - k_2 - \lambda x + \lambda k_2$$
$$= x - k_2 - \lambda k_1 + \lambda k_2$$
$$= x - (k_2 + \lambda k_1 - \lambda k_2)$$
$$= x - (\lambda k_1 + k_2(1-\lambda))$$
$$= x - (\lambda k_1 + (1-\lambda) k_2)$$

Since M is convex.

$\therefore k_1, k_2 \in M$

$\Rightarrow \lambda k_1 + (1-\lambda)k_2 \in M$

Thus,

$x - (\lambda k_1 + (1-\lambda)k_2) \in K$

Hence, K is convex.

Therefore $\exists$ a unique element k of the smallest norm. let this element be denoted by z.

Now if k is an element in M of norm

$$z - \langle z, k\rangle k \in K$$

and

$$\|z\|^2 \le \|z - \langle z,k\rangle k\|^2$$
$$= \langle z - \langle z,k\rangle k,\ z - \langle z,k\rangle k\rangle$$
$$= \langle z, z - \langle z,k\rangle k\rangle - \langle z,k\rangle\langle k, z - \langle z,k\rangle k\rangle$$
$$= \langle z,z\rangle - \langle \overline{z,k}\rangle\langle z,k\rangle - \langle z,k\rangle\langle \overline{z,k}\rangle + \langle z,k\rangle\langle\overline{z,k}\rangle\langle k,k\rangle$$

$$= \|z\|^2 - |\langle z,k\rangle|^2 - |\langle z,k\rangle|^2 + |\langle z,k\rangle|^2\|k\|^2$$

$$= \|z\|^2 - |\langle z,k\rangle|^2 - |\langle z,k\rangle|^2 + |\langle z,k\rangle|^2$$
$$= \|z\|^2 - |\langle z,k\rangle|^2$$

$\Rightarrow |\langle z,k\rangle|^2 \le 0$

$\Rightarrow \langle z,k\rangle = 0$

$\Rightarrow z \in M^\perp$

$\because z \in K$

$\Rightarrow \exists\ y \in M$ s.t.

$z = x - y$

$\Rightarrow \boxed{x = y + z}$ Hence Proved.

Uniqueness:- suppose that

$$x = y_1 + z_1 = y_2 + z_2$$

where $y_1, y_2 \in M$ and $z_1, z_2 \in M^\perp$

Then,

$$y_2 - y_1 = z_1 - z_2 \in M \cap M^\perp$$

Hence,

$$y_2 - y_1 = 0 \Rightarrow y_1 = y_2$$

similarly

$$z_1 = z_2$$

This proves the required uniqueness and Hence complete the proof of theorem.

8,9,10,11,14,16 * Riesz Representation Theorem

Statement:- Every bounded linear function f on a Hilbert space H can be represented in terms of the inner product namely

$$f(x) = \langle x, z\rangle$$

where z depends on f is uniquely determine by f and Hence the norm

$$\|z\| = \|f\|.$$

Proof :- we have to show that

(a) f has a representation

$$f(x) = \langle x, z \rangle \quad \text{--- (1)}$$

(b) z in (1) is unique.

(c) $\|z\| = \|f\|$ --- (2)

The details are as follows:

(a) If $f=0$ then (1) and (2) holds if we take $z=0$.

If $f \neq 0$.

first of all $z \neq 0$ since otherwise $f=0$ if,

$\langle x, z \rangle = 0 \ \forall \ x$ for which $f(x)=0$

i.e. $\langle x, z \rangle = 0 \quad \forall x \in \ker f$.

$\Rightarrow z \perp x \quad \forall x \in \ker f$

or we can say that $z \perp \ker f$.

Now, we consider the two cases.

Case I :- If $\ker f = H$ then,

$f(x) = 0 \quad \forall x \in H$

$\Rightarrow f(x) = \langle x, 0 \rangle \quad \forall x \in H$

Thus the result holds.

Case II :- If $\ker f \neq H$ then $\exists$ an element $z_0 \in H$ of unit norm orthogonal to $\ker f$.

since,

$z_0 \notin \ker f$

$\Rightarrow f(z_0) \neq 0$

$\therefore$ for any $x \in H$ we can write,

$$f\left(x - \frac{f(x)}{f(z_0)} z_0\right) = f(x) - \frac{f(x)}{f(z_0)} f(z_0)$$

$$= f(x) - f(x) = 0.$$

$$\Rightarrow x - \frac{f(x)}{f(z_0)} z_0 \in \ker f$$

Now, since z_0 is orthogonal to $\ker f$ so,

$$\left\langle x - \frac{f(x)}{f(z_0)} z_0, z_0 \right\rangle = 0$$

$$\langle x, z_0 \rangle - \frac{f(x)}{f(z_0)} \langle z_0, z_0 \rangle = 0$$

$$\Rightarrow \langle x, z_0 \rangle - \frac{f(x)}{f(z_0)} \|z_0\|^2 = 0$$

$$\Rightarrow \langle x, z_0 \rangle - \frac{f(x)}{f(z_0)} = 0$$

$$\Rightarrow f(z_0) \langle x, z_0 \rangle - f(x) = 0$$

$$\Rightarrow \langle x, \overline{f(z_0)} z_0 \rangle - f(x) = 0$$

$$\Rightarrow f(x) = \langle x, \overline{f(z_0)} z_0 \rangle$$

$$\Rightarrow f(x) = \langle x, z \rangle \quad \{\because z = \overline{f(z_0)} z_0\}$$

Page No.
Date: / /

(b) To prove uniqueness :- suppose that for all $x \in H$.

$f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$

$\Rightarrow \langle x, z_1 - z_2 \rangle = 0 \quad \forall x \in H.$

In particular choosing the particular element $x = z_1 - z_2$ then we have,

$\langle z_1 - z_2, z_1 - z_2 \rangle = 0$

$\Rightarrow \|z_1 - z_2\|^2 = 0$

$\Rightarrow z_1 - z_2 = 0$

$\Rightarrow z_1 = z_2$

(c) ~~We finally~~ If $f = 0$ then $z = 0$ and (2) holds. so let $f \neq 0$ then $z \neq 0$ so from (1) we have,

$f(x) = \langle x, z \rangle \quad \forall x \in H.$

putting $x = z$ we get

$f(z) = \langle z, z \rangle = \|z\|^2$

$\Rightarrow \|z\|^2 = f(z) \leq \|f\| \|z\| \quad \text{—— (3)}$

since $z \neq 0$ i.e

$\|z\| \neq 0$

Division by $\|z\|$ in (3) given us,

$\|z\| \leq \|f\| \quad \text{—— (4)}$

Page No.
Date: / /

It remains to show that

$\|z\| \geq \|f\|$

from (1) we have,

$f(x) = \langle x, z \rangle \quad \forall x \in H.$

$\Rightarrow |f(x)| = |\langle x, z \rangle|$

$\Rightarrow |f(x)| \leq \|x\| \|z\|$ {by cauchy Schwarz ineq

$\Rightarrow \dfrac{|f(x)|}{\|x\|} \leq \|z\|$

$\sup_{\|x\|=1} \dfrac{|f(x)|}{\|x\|} \leq \|z\|$

$\Rightarrow \sup\{|f(x)| : \|x\| = 1\} \leq \|z\|$

$\Rightarrow \|f\| \leq \|z\| \quad \text{—— (5)}$

combining (4) and (5) we get,

$\boxed{\|f\| = \|z\|}$ Hence Proved.

10/9/15

★ Bounded linear operators on Hilbert space :-

Theorem :- If $T \in B(X)$ then there exist a unique $U \in B(X)$ such that,

$\langle Tx, y \rangle = \langle x, Uy \rangle \quad \forall x, y \in X.$

Proof :- for a fixed $y \in X$ let $\phi : X \to C$ defined by,

$\phi(x) = \langle Tx, y\rangle \quad \forall x \in X$ —— ①

① To prove that ϕ is linear:-

for $\alpha, \beta \in \mathbb{C}$ and $x_1, x_2 \in X$ then

$$\phi(\alpha x_1 + \beta x_2) = \langle T(\alpha x_1 + \beta x_2), y\rangle$$
$$= \langle (T\alpha x_1 + T\beta x_2), y\rangle$$
$$= \langle T\alpha x_1, y\rangle + \langle T\beta x_2, y\rangle$$
$$= \alpha\langle Tx_1, y\rangle + \beta\langle Tx_2, y\rangle$$
$$= \alpha\phi(x_1) + \beta\phi(x_2)$$

Hence, ϕ is linear.

(ii) ϕ is bounded:-

$$\because \|\phi\| = \sup\{|\phi(x)| : \|x\| = 1\}$$
$$= \sup\{|\langle Tx, y\rangle| : \|x\| = 1\}$$
$$\leq \sup\{\|Tx\|\|y\| : \|x\| = 1\}$$
$$\leq \|y\| \sup\{\|T\|\|x\| : \|x\| = 1\}$$
$$\leq \|y\|\|T\|$$

so, $\|\phi\| \leq \|y\|\|T\|$

Hence ϕ is bounded.

consequently ϕ is a bounded linear functional on X.

$\therefore$ by Riesz-Representation theorem $\exists$ a unique z in X such that

$$\phi(x) = \langle x, z\rangle \quad \forall x \in X$$

Now define,

$$Uy = z$$

clearly U is linear

Then,

$$\phi(x) = \langle x, Uy\rangle$$

by eqn ①

$$\langle Tx, y\rangle = \langle x, Uy\rangle \quad \forall x, y \in X$$

putting $x = Uy$ we have,

$$\|Uy\|^2 = \langle Uy, Uy\rangle$$
$$= \langle TUy, y\rangle$$
$$\leq \|TUy\|\|y\|$$
$$\leq \|T\|\|y\|$$

$\Rightarrow \|Uy\|^2 \leq \|T\|\|y\|$

$\Rightarrow \|U\| \leq \|T\|$

Hence U is bounded.

To prove uniqueness:- suppose that there is another $V \in B(X)$ such that

$$\langle Tx, y\rangle = \langle x, Vy\rangle$$

also we have,

$$\langle Tx, y\rangle = \langle x, Uy\rangle$$

Thus we get,

$$\langle x, Uy\rangle = \langle x, Vy\rangle$$

$\Rightarrow \langle x, Uy - Vy\rangle = 0$

$\Rightarrow U, V \in B(X)$

$\therefore Uy, Vy \in X.$

Thus we can write,

Page No.
Date: / /

$\langle Uy - Vy, Uy - Vy \rangle = 0$

$\Rightarrow \|Uy - Vy\|^2 = 0$

$\Rightarrow Uy - Vy = 0$

$\Rightarrow \boxed{Uy = Vy}$

~~Thus U is linear~~.

Hence U is unique if $T \in B(X)$ then $\exists$ a unique $U \in B(X)$ s.t.

$\langle Tx, y \rangle = \langle x, Uy \rangle \quad \forall\, x, y \in X$ //

* **Adjoint Operator** :- let $T \in B(X)$. Then the adjoint of T, denoted by T^* is the unique element of $B(X)$ satisfying

$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall\, x, y \in X$

The adjoint operator of T^* is denoted by T^{**}.

Theorem :- The mapping $T \to T^*$ of $B(X)$ into itself has the following properties for $T, T_1, T_2 \in B(X)$ and $\alpha \in \mathbb{C}$ we have,

(i) $I^* = I$ where I is the identity operator.

(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$

(iii) $(\alpha T)^* = \bar{\alpha} T^*$

(iv) $(T_1 T_2)^* = T_2^* T_1^*$

(v) $T^{**} = (T^*)^* = T$

(vi) $\|T^*\| = \|T\|$

(vii) $\|T^*T\| = \|T\|^2$

(viii) If T is invertible, so is T^* and $(T^*)^{-1} = (T^{-1})^*$

Page No.
Date: / /

Proof :- (i) since $Ix = x \quad \forall\, x \in X$ we have,

$\langle Ix, y \rangle = \langle x, y \rangle$

and by the definition of I^*, we have.

$\langle Ix, y \rangle = \langle x, I^*y \rangle$

$\langle x, y \rangle = \langle x, I^*y \rangle$

$\langle x, y - I^*y \rangle = 0 \quad \forall\, x \in X$

$y - I^*y = 0$

$I^*y = y = Iy$

$I^* = I$

(ii) To prove that

$(T_1 + T_2)^* = T_1^* + T_2^*$

Now, for any $x, y \in X$

$\langle (T_1 + T_2)x, y \rangle = \langle T_1x + T_2x, y \rangle$

$= \langle T_1x, y \rangle + \langle T_2x, y \rangle$

$= \langle x, T_1^*y \rangle + \langle x, T_2^*y \rangle$

$= \langle x, T_1^*y + T_2^*y \rangle$

$= \langle x, (T_1^* + T_2^*)y \rangle$

$\Rightarrow \langle (T_1 + T_2)^* x, y \rangle = \langle x, (T_1^* + T_2^*)y \rangle$

$(T_1 + T_2)^* = T_1^* + T_2^*$

(iii) $(\alpha T)^* = \bar{\alpha} T^*$

for $x, y \in X$ and $\alpha \in \mathbb{C}$

$\langle \alpha Tx, y \rangle = \alpha \langle Tx, y \rangle$

$= \alpha \langle x, T^* y \rangle$

$= \langle x, \bar{\alpha} T^* y \rangle$

Hence, $(\alpha T)^* = \bar{\alpha} T^*$

(iv) To prove that

$(T_1 T_2)^* = T_2^* T_1^*$

for $x, y \in X$

$\langle (T_1 T_2) x, y \rangle = \langle (T_1 T_2) x, y \rangle$

$= \langle T_1 T_2 (x), y \rangle$

$= \langle T_1 x, T_2^* y \rangle$

$= \langle x_1, T_2^* (T_1^* (y)) \rangle$

$\langle (T_1 T_2) x, y \rangle = \langle x, T_2^* T_1^* (y) \rangle$

$\Rightarrow (T_1 T_2)^* = T_2^* T_1^*$

(v) To prove that

$T^{**} = (T^*)^* = T$

for any $x, y \in X$.

$\langle x, T^{**} y \rangle = \langle T^* x, y \rangle$

$= \overline{\langle y, T^* x \rangle}$

$= \overline{\langle Ty, x \rangle}$

$= \langle x, Ty \rangle$

$T^{**} = (T^*)^* = T$

(vi) To prove that $\|T^*\| = \|T\|$

for any $x, y \in X$.

$\Rightarrow \|T^* x\|^2 = \langle T^* x, T^* x \rangle$

$\leq |(T(T^* x), x)|$

$\leq \|T(T^* x)\| \, \|x\|$

$\leq \|T\| \, \|T^*(x)\| \, \|x\|$

$\Rightarrow \|T^*(x)\|^2 \leq \|T\| \, \|T^*(x)\| \, \|x\|$

$\Rightarrow \|T^*(x)\| \leq \|T\| \, \|x\|$ —— ①

Thus $\|T^*\| \leq \|T\|$

Now,

$\|T^{**}\| \leq \|T^*\|$ $\{\because T = T^*\}$

$\Rightarrow \|T^{**}\| \leq \|T^*\|$

$\Rightarrow \|T\| \, \|x\| \leq \|T^*(x)\|$ —— ②

from ① and ② we get,

$\|T^*(x)\| = \|T\| \, \|x\|$.

$$\boxed{\|T^*\| = \|T\|}$$

(vii) To prove that,

$\|T^* T\| = \|T\|^2$

for $T_1, T_2 \in B(X)$ and $\alpha \in X$ we have,

$\|(T_1 T_2) x\| = \|T_1 T_2 (x)\|$

$\leq \|T_1\| \, \|T_2 (x)\|$

$\|T_1 T_2\| \, \|x\| \leq \|T_1\| \, \|T_2\| \, \|x\|$

$\|T_1 T_2\| \leq \|T_1\| \, \|T_2\|$

It follows that,

$$\|T^*T\| \le \|T^*\|\|T\|$$
$$\le \|T\|\|T\|$$
$$\le \|T\|^2 \quad \text{--- (1)}$$

again for $x \in X$:

$$\|Tx\|^2 = \langle Tx, Tx\rangle$$
$$= \langle x, T^*(T(x))\rangle$$
$$\le \|x\|\,\|T^*(T(x))\|$$
$$\le \|x\|\,\|T^*T\|\|x\|.$$
$$\|Tx\|^2 \le \|x\|^2\,\|T^*T\|$$
$$\|Tx\| \le \|x\|\,\|T^*T\|^{1/2}$$
$$\|T\| \le \|T^*T\|^{1/2}$$
$$\|T\|^2 \le \|T^*T\| \quad \text{--- (2)}$$

so, by (1) and (2) we get,

$$\|T^*T\| = \|T\|^2$$

(viii) since $I^* = I$ and
$TT^{-1} = T^{-1}T = I$

we have,

$$(TT^{-1})^* = I^* = I$$

$$(TT^{-1})^* = (T^{-1})^* T^*$$

$\because (T^{-1})^*T^* = I$ and so,

$$(T^*)^{-1} = (T^{-1})^*$$

This complete the proof.

* **Definition:-** let $T \in B(X)$ then,

(i) T is self-adjoint or Hermition if $T = T^*$

(ii) T is normal if $T^*T = TT^*$.

(iii) T is unitary if $T^*T = TT^* = I$.

(iv) T is positive if $\langle Tx, x\rangle \ge 0 \ \forall \ x \in X$.
It is called strictly positive if $\langle Tx, T\rangle = 0$ only if $x = 0$.

* **Reflexive Hilbert space**

Theorem:- Every Hilbert space is reflexive.

Proof:- let H be a Hilbert space and define a mapping from.

$$T: H^* \to H \quad \text{--- (1)}$$
$$T(f) = y \ \forall \ f \in H^* \quad \text{--- (2)}$$

where y is the representer of f i.e.

$$f(x) = \langle x, y\rangle \ \forall \ x \in H \quad \text{--- (3)}$$

First we shall show that H^* is Hilbert space so for $f, g \in H^*$ take.

$$\langle f, g\rangle = \langle Tg, Tf\rangle$$
$$= \langle z, y\rangle \ge 0 \quad \text{--- (4)}$$

$\Rightarrow \langle f, g\rangle = \langle z, y\rangle \geq 0$

with $g(x) = \langle x, z\rangle \ \forall\ x \in H$.

$\Rightarrow \langle f, g\rangle \geq 0$

also,

$\langle f, f\rangle = 0 \Leftrightarrow \langle Tf, Tf\rangle = 0$

$\Rightarrow Tf = 0$

$\Rightarrow f = 0$

$\Rightarrow \langle f, f\rangle = 0$

$\Rightarrow f = 0$

$\langle f+g, h\rangle = \langle Th, T(f+g)\rangle$

$= \langle Th, Tf\rangle + \langle Th, Tg\rangle$

$= \cancel{\langle Th, Tf}$

$= \langle f, h\rangle + \langle g, h\rangle$

and for any scalar α and,

$\langle \alpha f, g\rangle = \langle Tg, T(\alpha f)\rangle$

$= \langle Tg, \bar{\alpha} T(f)\rangle$

$= \alpha \langle Tg, Tf\rangle$

$= \alpha \langle f, g\rangle$

Thus $\langle f, g\rangle$ given by (4) is a inner product for H^*.

we know that X^* is complete norm linear space with norm induced by inner product.

Hence H^* is Hilbert space.

Now we shall show that the canonical embedding $\Pi: H \to H^{**}$ is surjective.

let $F \in H^{**}$ and its Riesz representation be given by,

$F(f) = \langle f, f_0\rangle$

$= \langle Tf_0, Tf\rangle$ —— (4)

put $Tf_0 = x_1$ so that we have,

$\langle Tf_0, Tf\rangle = \langle x, y\rangle$

$= f(x)$ —— (5)

$F(f) = f(x) \ \forall\ f \in H^*$

Hence $\boxed{\Pi(x) = F}$ Hence Proved.

* **Hilbert Space:-** A complete inner product space is called Hilbert space.

14* **Theorem:** let X be a Hilbert space and Y be a proper closed subspace of X then $\exists$ a non-zero vector $x \in X$ s.t. $x \perp Y$.

Proof:- {unit III $x = z_0$, $Y = M$} Repeat.

* **Dual of the Hilbert Space:-** If X is Hilbert space then set of all bounded linear functional on X is called dual of X and it is denoted by X^*.

* **Theorem:-** for $x \in X$ let f_x be a functional defined by $f_x(y) = \langle y, x \rangle$ $\forall y \in X$ then f_x is linear and bounded i.e. $f_x \in X^*$ also $\|f_x\| = \|x\|$.

Proof:- let $y_1, y_2 \in X$ and a, b are scalars. then,

$$f_x(ay_1 + by_2) = \langle ay_1 + by_2, x \rangle \quad \{\text{by def}^n \text{ of } f_x\}$$

$$= \langle ay_1, x \rangle + \langle by_2, x \rangle$$
$$= a\langle y_1, x \rangle + b\langle y_2, x \rangle$$

$$f_x(ay_1 + by_2) = a\, f_x(y_1) + b\, f_x(y_2)$$

$\Rightarrow f_x$ is linear.

Also,

$$\|f_x(y)\| = |\langle y, x \rangle|$$
$$\leq \|y\| \cdot \|x\| \quad \{\text{by Cauchy Schwarz inequality}\}$$

$$\Rightarrow \frac{\|f_x(y)\|}{\|y\|} \leq \|x\|$$

$$\Rightarrow \sup \frac{\|f_x(y)\|}{\|y\|} \leq \|x\|$$

$$\Rightarrow \|f_x\| \leq \|x\| \quad \text{---(1)}$$

since

$$\|f_x\| = \sup_x \frac{\|f_x(x)\|}{\|x\|} \quad \text{we get}$$

$$\|f_x\| \geq \|x\| \quad \text{---(2)}$$

inequalities (1) and (2) implies

$$\|f_x\| = \|x\|$$

Hence, f_x is linear and bounded i.e. $f_x \in X^*$ also $\|f_x\| = \|x\|$.

13, 14 * **Theorem:-** let X be a Hilbert and let the mapping $\psi: X \to X^*$ be defined by $\psi(x) = f_x$ $\forall x \in X$ where $f_x(y) = \langle y, x \rangle$ $\forall y \in X$. Then ψ is one-one, onto conjugate linear norm preserving and Hence isomorphic.

Proof:- given that X be a Hilbert space and let the mapping $\psi: X \to X^*$ be defined by

$$\psi(x) = f_x \quad \forall x \in X \text{ --- (1)}$$

where

$$f_x(y) = \langle y, x \rangle \quad \forall y \in X \text{ --- (2)}$$

(i) for linearly :-

$$\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)$$

let $x_1, x_2 \in X$ then by (1) we get

$$\psi(x_1 + x_2) = f_{x_1 + x_2}$$

Now,

$$f_{x_1+x_2}(y) = \langle y, x_1 + x_2 \rangle$$
$$= \langle y, x_1 \rangle + \langle y, x_2 \rangle$$
$$= f_{x_1}(y) + f_{x_2}(y)$$

$f_{x_1+x_2}(y) = (f_{x_1}+f_{x_2})(y)$

$\therefore f_{x_1+x_2} = f_{x_1}+f_{x_2}$

Hence

$\psi(x_1+x_2) = f_{x_1+x_2}$

$= f_{x_1}+f_{x_2}$

$\psi(x_1+x_2) = \psi(x_1)+\psi(x_2)$

also if $\alpha \in C$

$f_{\alpha x}(y) = \langle y, \alpha x\rangle$

$= \bar{\alpha}\langle y, x\rangle$

$f_{\alpha x}(y) = \bar{\alpha} f_x(y) \quad \forall y \in X$

$f_{\alpha x} = \bar{\alpha} f_x$

Hence

$\psi(\alpha x) = f_{\alpha x} = \bar{\alpha} f_x = \bar{\alpha}\psi(x)$

$\Rightarrow \psi$ is conjugate linear.

(ii) For one-one:- let $x_1, x_2 \in X$ s.t.

$\psi(x_1) = \psi(x_2)$

$f_{x_1} = f_{x_2}$

$f_{x_1}(y) = f_{x_2}(y)$

$\langle y, x_1\rangle = \langle y, x_2\rangle$

$\langle y, x_1\rangle - \langle y, x_2\rangle = 0$

$\langle y, x_1 - x_2\rangle = 0$

if $y = x_1 - x_2$ then

$\langle x_1 - x_2, x_1 - x_2\rangle = 0$

$\|x_1 - x_2\|^2 = 0$

$\|x_1 - x_2\| = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

i.e. $\psi(x_1) = \psi(x_2)$

$\Rightarrow x_1 = x_2$

$\Rightarrow \psi$ is one-one.

(iii) Onto:- let f be an arbitrary function in X^* then by Riesz representation theorem $\exists$ a unique $y \in X$ s.t.

$f(x) = \langle x, y\rangle \quad \forall x \in X$

then by def^n of f_y we have,

$f_y(x) = \langle x, y\rangle = f(x)$

i.e. $f_y(x) = f(x) \quad \forall x \in X.$

$\Rightarrow f_y = f$

Now $\psi(y) = f_y = f$

Thus for each $f \in X^*$ we get $y \in X^*$ s.t.

$\psi(y) = f_y \quad \forall y.$

$\Rightarrow \psi$ is onto.

(iv) Norm Preserving:-

$\|f_x - f_y\| = \|f_{x-y}\|$

$= \|x-y\|$

$\therefore \|\psi(x) - \psi(y)\| = \psi\|x-y\|$

$\Rightarrow \psi$ is norm preserving.

$\because \psi$ is one-one, onto and preserve the norm $\therefore$ therefore ψ is isometric from X onto X^*.

15 * Theorem:- The dual of Hilbert space is Hilbert space.

Proof:- let X be a Hilbert space then every point x^* in X^* is of the form. $f_x \ \forall x \in X$ where

$$f_x(u) = \langle u, x\rangle \quad \forall u \in X.$$

Thus we can defined the inner product in X^* by

$$\langle x^*, y^*\rangle = \langle f_x, f_y\rangle = \langle y, x\rangle \quad \text{--- (1)}$$

where $x^* = f_x$ and $y^* = f_y$ are any two points in X^*

(i) $\langle x^*, x^*\rangle \geq 0$

$$\langle x^*, x^*\rangle = \langle f_x, f_x\rangle$$
$$= \langle x, x\rangle \geq 0$$
$$= \|x\|^2 \geq 0$$

$\langle x^*, x^*\rangle \geq 0.$

(ii) let $x^*, y^*, z^* \in X^*$

$$\langle \alpha x^* + \beta y^*, z^*\rangle = \alpha\langle x^*, z^*\rangle + \beta\langle y^*, z^*\rangle$$

$$\langle \alpha x^* + \beta y^*, z^*\rangle = \langle \alpha f_x + \beta f_y, f_z\rangle$$
$$= \langle f_{\bar{\alpha} x} + f_{\bar{\beta} y}, f_z\rangle$$
$$= \langle f_{\bar{\alpha} x + \bar{\beta} y}, f_z\rangle$$
$$= \langle z, \bar{\alpha} x + \bar{\beta} y\rangle$$
$$= \alpha\langle z, x\rangle + \beta\langle z, y\rangle$$
$$= \alpha\langle f_x, f_z\rangle + \beta\langle f_y, f_z\rangle$$
$$= \alpha\langle x^*, z^*\rangle + \beta\langle y^*, z^*\rangle$$

i.e.

$$\langle \alpha x^* + \beta y^*, z^*\rangle = \alpha\langle x^*, z^*\rangle + \beta\langle y^*, z^*\rangle$$

(iii) $\overline{\langle x^*, y^*\rangle} = \langle y^*, x^*\rangle$

$$\overline{\langle x^*, y^*\rangle} = \overline{\langle f_x, f_y\rangle}$$
$$= \overline{\langle y, x\rangle}$$
$$= \langle x, y\rangle$$
$$= \langle f_y, f_x\rangle$$
$$\overline{\langle x^*, y^*\rangle} = \langle y^*, x^*\rangle$$

(iv) $\langle x^*, x^*\rangle = 0 \Leftrightarrow x^* = 0$

$$\langle x^*, x^*\rangle = 0 \Leftrightarrow \langle f_x, f_x\rangle = 0$$
$$\Leftrightarrow \langle x, x\rangle = 0$$
$$\Leftrightarrow x = 0$$
$$\Leftrightarrow \langle y, x\rangle = 0$$
$$\Leftrightarrow f_x(y) = 0 \quad \forall y.$$
$$\Leftrightarrow f_x = 0$$

This is an inner product and Hence X^* is an inner product space.

16 * Theorem:- A Hilbert space is reflexive.

Proof:- since Hilbert space is a uniformly convex Banach space so it is sufficient to prove that every convex banach space is reflexive.

we know that X is reflexive $\Leftrightarrow$ $f \in X^*$ attains its norm on the ~~unit~~ unit ball also $f \in X^*$ attaince its norm.

The unit Ball is equivalent to every $f \in S^*$ attains its norm on S where,

$S = \{x \in X : \|x\| = 1\}$ and
$S^* = \{f \in X^* : \|f\| = 1\}$
as if it is true for S^* then if $f \in S^*$ and $g \in X^*$. let
$$f = \frac{g}{\|g\|}$$
then, $\|f\| = 1$
$\because f \in S^*$
$\Rightarrow \exists x_0$ s.t.
$$f(x_0) = \|f\| = 1$$
$$f(x_0) = \frac{g(x_0)}{\|g\|}$$
$$1 = \frac{g(x_0)}{\|g\|} \Rightarrow \|g\| = g(x_0)$$
It is true for X^*

To proving this we shall prove the following lemma.

Lemma:- If x is uniformly convex $f \in S^*$ and $\{x_n\}$ is a sequence in S s.t. $f(x_n) \to 1$ then $\{x_n\}$ is a cauchy sequence

Proof of lemma:- if $\{x_n\}$ is not cauchy sequence then for any $\varepsilon > 0$ there is a subsequence $\{x_{n_J}\}$ s.t.
$$\|x_{n_J} - x_{n_K}\| \geq \varepsilon \quad \forall \; J, K, \; J \neq K.$$

By uniformly convexity it follows that for $J \neq K$ $\exists \; \delta > 0$ s.t.
$$\left\| \frac{x_{n_J} + x_{n_K}}{2} \right\| \leq 1 - \delta$$
But then,
$$\left| f\left(\frac{x_{n_J} + x_{n_K}}{2} \right) \right| \leq \|f\| \left\| \frac{x_{n_J} + x_{n_K}}{2} \right\| \leq 1 - \delta$$
i.e.
$$\left| f\left(\frac{x_{n_J} + x_{n_K}}{2} \right) \right| \leq 1 - \delta.$$
$$\left| \frac{1}{2} \left(f(x_{n_J} + x_{n_K}) \right) \right| \leq 1 - \delta$$

Thus taking $J, K \to \infty$
$\Rightarrow \frac{1}{2}(1 + 1) \leq 1 - \delta$
$\Rightarrow \quad 1 \leq 1 - \delta$
which is a contradiction.
$\Rightarrow \{x_n\}$ is cauchy sequence.

Proof of the main theorem. let $f \in S^*$ and $\{x_n\}$ a sequence in S s.t.
$$f(x_n) \to 1$$
then by above lemma $\{x_n\}$ is a cauchy ~~theorem.~~ sequence.
$\because$ space is complete Therefore $\exists$ a point $x \in S$. $x_n \to x$.

$\Rightarrow f(x_n) \to f(x)$

Thus,

$$f(x) = 1$$

since convergent limit is unique i.e.

$$f(x) = \|f\|$$

$\Rightarrow X$ is reflexive.